

ON THE AUTOMORPHISMS AND REPRESENTATIONS OF POLYADIC GROUPS

H. KHODABANDEH AND M. SHAHRYARI

ABSTRACT. Using a unified method, we determine the structure of automorphisms and representations of arbitrary polyadic groups. More precisely, for a polyadic group $(G, f) = \text{der}_{\theta, b}(G, \cdot)$, we obtain a complete description of automorphisms and representations of (G, f) in terms of automorphisms and representations of the binary group (G, \cdot) .

1. INTRODUCTION

A non-empty set G together with an n -ary operation $f : G^n \rightarrow G$ is called an n -ary *quasi-group* or a *polyadic quasi-group*, if for all $x_0, x_1, \dots, x_n \in G$ and for any fixed $i \in \{1, \dots, n\}$, there exists a unique element $y \in G$, such that

$$f(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) = x_0.$$

In the binary case (i.e., for $n = 2$), a polyadic quasi-group is just usual quasi-group. If the operation f is also associative, then we call (G, f) an n -ary *group* or a *polyadic group*.

According to a general convention used in the theory of n -ary systems, the sequence of elements x_i, x_{i+1}, \dots, x_j is denoted by x_i^j . In the case $j < i$ it is the empty symbol. If $x_{i+1} = x_{i+2} = \dots = x_{i+t} = x$, then instead of x_{i+1}^{i+t} we write $x_i^{(t)}$. In this convention $f(x_1, \dots, x_n) = f(x_1^n)$ and

$$f(x_1, \dots, x_i, \underbrace{x, \dots, x}_t, x_{i+t+1}, \dots, x_n) = f(x_1^i, x_i^{(t)}, x_{i+t+1}^n).$$

As an example of a polyadic quasi-group, suppose (G, \cdot) is an ordinary quasi-group (so the equations $ax = b$ and $xa = b$ have unique solutions for x). Let $\alpha_1, \dots, \alpha_n$ be arbitrary automorphisms of (G, \cdot) and $b \in G$ be fixed. If we define

$$f(x_1^n) = \alpha_1(x_1) \dots \alpha_n(x_n)b,$$

then (G, f) becomes an n -ary quasi-group. A polyadic quasi-group of this type, is called a *linear polyadic quasi-group*. As a special case, if (G, \cdot) is

Date: November 30, 2010.

MSC(2010): 20N15

Keywords: Polyadic groups; Polyadic quasi-groups; Homotopy; Autotopy; Automorphisms of polyadic groups; Representations; Retract group.

an ordinary group, $\alpha_1 = \dots = \alpha_n = \text{identity}$ and $b = 1$, then we have

$$f(x_1^n) = x_1 x_2 \dots x_n,$$

and (G, f) is an n -ary group which is called the n -ary group, derived from (G, \cdot) and it is denoted by $\text{der}^n(G, \cdot)$. More generally, suppose (G, \cdot) is an ordinary group and θ is an automorphism with a fixed point $b \in G$, such that $\theta^{n-1}(x) = bxb^{-1}$ for all $x \in G$. Now, define

$$f(x_1^n) = x_1 \theta(x_2) \theta^2(x_3) \dots \theta^{n-1}(x_n) b.$$

It can be shown that (G, f) is an n -ary group which we denote it by $\text{der}_{\theta, b}(G, \cdot)$ and we call it (θ, b) -derived polyadic group from (G, \cdot) . It is proved that the converse is also true, i. e. every polyadic group can be uniquely expressed as $(G, f) = \text{der}_{\theta, b}(G, \cdot)$. This is the content of Hosszú-Gluskin Theorem, which is formulated as follows.

Theorem 1.1. *Let (G, f) be an n -ary group. Then*

- (1) *on G one can define an operation \cdot such that (G, \cdot) is a group,*
- (2) *there exist an automorphism θ of (G, \cdot) and $b \in G$, such that $\theta(b) = b$,*
- (3) *$\theta^{n-1}(x) = bxb^{-1}$, for every $x \in G$,*
- (4) *$f(x_1^n) = x_1 \theta(x_2) \theta^2(x_3) \dots \theta^{n-1}(x_n) b$, for all $x_1, \dots, x_n \in G$.*

For a proof, see [5] or [6].

From the definition of an n -ary group (G, f) we can directly see that for every $x \in G$, there exists only one $y \in G$ satisfying the equation

$$f(\overset{(n-1)}{x}, y) = x.$$

This element is called *skew* to x and is denoted by \overline{x} . In a ternary group ($n = 3$) derived from a binary group (G, \cdot) , the skew element coincides with the inverse element in (G, \cdot) . Thus, in some sense, the skew element is a generalization of the inverse element in binary groups. Nevertheless, the concept of skew elements plays a crucial role in the theory of n -ary groups. Namely, as Dörnte proved, the following theorem (see [2]).

Theorem 1.2. *In any n -ary group (G, f) the following identities*

$$f(\overset{(i-2)}{x}, \overline{x}, \overset{(n-i)}{x}, y) = f(y, \overset{(n-j)}{x}, \overline{x}, \overset{(j-2)}{x}) = y,$$

$$f(\overset{(k-1)}{x}, \overline{x}, \overset{(n-k)}{x}) = x$$

hold for all $x, y \in G$, $2 \leq i, j \leq n$ and $1 \leq k \leq n$.

Fixing in an n -ary operation f all inner elements a_2, \dots, a_{n-1} we obtain a new binary operation

$$x \cdot y = f(x, a_2^{n-1}, y).$$

Such obtained groups (G, \cdot) is called a *retract* of (G, f) . Choosing different elements a_1, \dots, a_{n-1} we obtain different retracts. Retracts of a fixed n -ary group are isomorphic (cf. [6]). So, we can consider only retracts of the form

$$x \cdot y = f(x, \overset{(n-2)}{a}, y).$$

Such retracts will be denoted by $Ret_a(G, f)$. The identity of the group $Ret_a(G, f)$ is \bar{a} . One can verify that the inverse element to x has the form

$$x^{-1} = f(\bar{a}, \overset{(n-3)}{x}, \bar{x}, \bar{a}).$$

The binary group, which we said about in the theorem 1.1, is in fact $Ret_a(G, f)$ and the automorphism θ is defined as $\theta(x) = (\bar{a}, x, \overset{(n-2)}{a})$.

Binary retracts of an n -ary group (G, f) are commutative only in the case when there exists an element $a \in G$ such that

$$f(x, \overset{(n-2)}{a}, y) = f(y, \overset{(n-2)}{a}, x)$$

holds for all $x, y \in G$. An n -ary group with this property is called *semiaabelian*. It satisfies the identity

$$f(x_1^n) = f(x_n, x_2^{n-1}, x_1)$$

(see [4]).

One can prove (cf. [8]) that a semiabelian n -ary group is *medial*, i.e., it satisfies the identity

$$f(f(x_{11}^{1n}), f(x_{21}^{2n}), \dots, f(x_{n1}^{nn})) = f(f(x_{11}^{n1}), f(x_{12}^{n2}), \dots, f(x_{1n}^{nn})).$$

In such n -ary groups

$$\overline{f(x_1^n)} = f(\bar{x}_1, \bar{x}_2, \bar{x}_3, \dots, \bar{x}_n)$$

for all $x_1, \dots, x_n \in G$.

The idea of investigations of such polyadic group goes back to E. Kasner's lecture [9] at the fifty-third annual meeting of the American Association for the Advancement of Science in 1904. But the first paper concerning the theory of n -ary groups was written (under inspiration of Emmy Noether) by W. Dörnte in 1928 (see [2]). In 1940 E. L. Post published an extensive study of n -groups in which the well-known Post's Coset Theorem appeared, see [11].

Representation theory of polyadic groups is investigated by W. Dudek and M. Shahryari in [7]. In [12], the second author, studied characters of finite polyadic groups. In this article, using a unified method (homotopies and autotopies of polyadic groups), we study the structure of automorphisms and representations of polyadic group. More precisely, if $(G, f) = der_{\theta, b}(G, \cdot)$, we express an automorphism (a representation) of (G, f) , as a product of a certain automorphism (representation) of the binary group (G, \cdot) and a translation.

2. AUTOMORPHISMS

Suppose (G, f) and (H, h) are two n -ary quasi-groups. Suppose there are maps $\alpha_1, \dots, \alpha_{n+1}$ from G to H , such that

$$\alpha_{n+1}(f(x_1, \dots, x_n)) = h(\alpha_1(x_1), \dots, \alpha_n(x_n)).$$

Then we say that $T = (\alpha_1, \dots, \alpha_{n+1})$ is a *homotopy* from (G, f) to (H, h) . If all maps, α_i , are bijections, then T is an *isotopy*. We call T an *autotopy* of (G, f) , if $(G, f) = (H, h)$ and T is an isotopy. The set of all autotopies of an n -ary quasi-group (G, f) is denoted by $\mathfrak{T}(G, f)$. If we define a binary operation

$$(\alpha_1, \dots, \alpha_{n+1}) \circ (\beta_1, \dots, \beta_{n+1}) = (\alpha_1\beta_1, \dots, \alpha_{n+1}\beta_{n+1}),$$

then $\mathfrak{T}(G, f)$ becomes a group. In general, if $T = (\alpha_1, \dots, \alpha_{n+1})$ is a homotopy from (G, f) to (H, h) and $S = (\beta_1, \dots, \beta_{n+1})$ is another homotopy from (K, g) to (H, h) , then we define their composition as

$$(\alpha_1, \dots, \alpha_{n+1}) \circ (\beta_1, \dots, \beta_{n+1}) = (\alpha_1\beta_1, \dots, \alpha_{n+1}\beta_{n+1}).$$

It is easy to see that if $T = (\alpha_1, \dots, \alpha_{n+1})$ is an isotopy from (G, f) to (H, h) , then we have

$$\mathfrak{T}(G, f) = T^{-1} \circ \mathfrak{T}(H, h) \circ T.$$

A map $\psi : G \rightarrow H$ is a *homomorphism* from (G, f) to (H, h) , if $T = (\psi, \dots, \psi)$ is a homotopy. If

$$T = (\psi, \dots, \psi) \in \mathfrak{T}(G, f),$$

then we say that ψ is an automorphism of (G, f) . For example, let $(G, f) = \text{der}_{\theta, b}(G, \cdot)$. Then for any $a \in G$, the map $\psi(x) = a\theta(xa^{-1})b^{-1}$ is an automorphism of (G, f) . The group of automorphisms of an arbitrary (G, f) , will be denoted by $\text{Aut}(G, f)$.

Let (G, \cdot) be an ordinary quasi-group and $a \in G$. We define the *left* and the *right translation maps*, L_a and R_a by $L_a(x) = ax$ and $R_a(x) = xa$, respectively. If G is a group, we denote by I_a , the inner automorphism $I_a = (x)a^{-1}xa$. For the proof of the next theorem, see [10].

Theorem 2.1. *Suppose (G, \cdot) is an ordinary group and $(G, f) = \text{der}^n(G, \cdot)$. Then T is an autotopy of (G, f) , if and only if*

$$T = (L_{a_1}I_{a_1}, L_{a_2}I_{a_1a_2}, \dots, L_{a_n}I_{a_1\dots a_n}, R_{a_1\dots a_n}) \circ (\phi, \dots, \phi),$$

where $a_1^n \in G$ and $\phi \in \text{Aut}(G, \cdot)$. Further, the above decomposition is unique.

Now, we are going to determine the structure of automorphism of a polyadic group of type $(G, f) = \text{der}_{\theta}(G, \cdot)$, (i.e., $b = 1$). The general case, will be proved later. In what follows, ε denotes the identity map.

Theorem 2.2. *Let $(G, f) = \text{der}_{\theta}(G, \cdot)$, where (G, \cdot) is an ordinary group and θ is an ordinary automorphism with $\theta^{n-1} = \varepsilon$. Then*

$$\text{Aut}(G, f) = \{R_a\phi : \bar{a} = a, \phi \in \text{Aut}(G, \cdot), [\theta, \phi] = I_a\},$$

where the bracket denotes the ordinary commutator $\theta\phi\theta^{-1}\phi^{-1}$.

Proof. Define a second operation g on G by $g(x_1^n) = x_1x_2\ldots x_n$. Now, $T = (\varepsilon, \theta, \theta^2, \ldots, \theta^{n-2}, \varepsilon, \varepsilon)$ is an isotopy between (G, f) and (G, g) . So we have

$$\mathfrak{T}(G, f) = T^{-1} \circ \mathfrak{T}(G, g) \circ T.$$

Using 2.1, we have also,

$$\begin{aligned} \mathfrak{T}(G, g) &= \{(L_{a_1}I_{a_1}, L_{a_2}I_{a_1a_2}, \ldots, L_{a_n}I_{a_1\ldots a_n}, R_{a_1\ldots a_n}) \circ (\phi, \ldots, \phi) \\ &\quad : a_1^n \in G, \phi \in \text{Aut}(G, \cdot)\}. \end{aligned}$$

Hence, every element of $\mathfrak{T}(G, f)$ can be written uniquely as

$$(L_{a_1}I_{a_1}\phi, \theta^{-1}L_{a_2}I_{a_1a_2}\phi\theta, \ldots, \theta^{1-n}L_{a_n}I_{a_1\ldots a_n}\phi\theta^{n-1}, R_{a_1\ldots a_n}\phi),$$

where $a_1^n \in G$ and $\phi \in \text{Aut}(G, \cdot)$. It is clear that $\text{Aut}(G, f)$ is precisely the set of all ψ 's such that $(\psi, \ldots, \psi) \in \mathfrak{T}(G, f)$. So, we must determine all elements of $\mathfrak{T}(G, f)$, with the equal entries. Therefore, suppose we have

$$\begin{aligned} \psi &= L_{a_1}I_{a_1}\phi \\ &= \theta^{-1}L_{a_2}I_{a_1a_2}\phi\theta \\ &\vdots \\ &= \theta^{2-n}L_{a_{n-1}}I_{a_1\ldots a_{n-1}}\phi\theta^{n-2} \\ &= L_{a_n}I_{a_1\ldots a_n}\phi \\ &= R_{a_1\ldots a_n}\phi. \end{aligned}$$

Suppose also $a = a_1$. Then, clearly $\psi = R_a\phi$, thus we prove that $\bar{a} = a$ and $[\theta, \phi] = I_a$. We, have the following steps.

i- The equality of the first and the last entry ($R_a\phi = R_{aa_2\ldots a_n}\phi$), implies $a_2a_3\cdots a_n = 1$.

ii- The equality of the first and n -th entry ($R_a\phi = L_{a_n}I_{aa_2\ldots a_n}\phi$), implies $a_n = a$.

iii- Now, we use the equality of the second and the third entries. We have $R_a\phi = \theta^{-1}L_{a_2}I_{aa_2}\phi\theta$, so for all x ,

$$\begin{aligned} \phi(x)a &= \theta^{-1}(a_2a_2^{-1}a^{-1}\phi(\theta(x))aa_2) \\ &= \theta^{-1}(a^{-1})\phi(\theta(x))\theta^{-1}(a)\theta^{-1}(a_2). \end{aligned}$$

Hence, if we put $x = 1$, then we obtain $\theta(a) = a_2$.

iv- Continuing this way, using other equalities, we conclude that for all i , $a_i = \theta^{i-1}(a)$. Hence

$$\begin{aligned} f\binom{(n)}{a} &= a\theta(a)\ldots\theta^{n-1}(a) \\ &= aa_2\ldots a_n \\ &= a. \end{aligned}$$

So $\bar{a} = a$. Now, we have $a\theta(a)\theta^2(a)\dots\theta^{n-2}(a) = 1$, hence applying the equality of the first and $(n-2)$ -th entries, we obtain

$$\begin{aligned} R_a\phi(x) &= \theta^{2-n}L_{\theta^{n-2}(a)}I_{a\theta(a)\dots\theta^{n-2}(a)}\phi\theta^{n-2}(x) \\ &= \theta^{2-n}L_{\theta^{n-2}(a)}\phi\theta^{n-2}(x) \\ &= \theta^{2-n}(\theta^{n-2}(a)\phi(\theta^{n-2}(x))) \\ &= a\theta^{2-n}(\phi(\theta^{n-2}(x))). \end{aligned}$$

So, we have $\theta^{2-n}\phi\theta^{n-2}(x) = a^{-1}\phi(x)a$. But, $\theta^{n-2} = \theta^{-1}$, hence we have

$$\theta\phi\theta(x) = a^{-1}\phi(x)a,$$

which is equivalent to $[\theta, \phi] = I_a$.

Conversely, suppose $a \in G$ has the property $\bar{a} = a$. Let $\phi \in \text{Aut}(G, \cdot)$ such that $[\theta, \phi] = I_a$. We prove that $R_a\phi$ is an automorphism of (G, f) . Since $\bar{a} = a$, we know that the following $(n+1)$ -tuple is an autotopy of (G, f) ;

$$(R_a\phi, \theta^{-1}L_{\theta(a)}I_{a\theta(a)}\phi\theta, \dots, \theta^{1-n}L_{\theta^{n-1}(a)}I_a\phi\theta^{n-1}, R_a\phi).$$

It is enough to show that all entries of the above autotopy are equal. Using $[\theta, \phi] = I_a$, we obtain

$$\theta^{-i}\phi\theta^i(x) = \theta^{-1}(a)\dots\theta^{-1}(a)\phi(x)\theta^{-1}(a^{-1})\dots\theta^{-i}(a^{-1}),$$

for all i . A few computations show that the above relation is equivalent to

$$R_a\phi = \theta^{-i}L_{\theta^i(a)}I_{a\theta(a)\dots\theta^i(a)}\phi\theta^i.$$

Hence, all entries of our autotopy are equal and so $R_a\phi$ is an automorphism.

Definition 2.3. An element $a \in G$ with the property $\bar{a} = a$ is called an *idempotent*. If $(G, f) = \text{der}_\theta(G, \cdot)$, then the set of all idempotents which are also in the center of (G, \cdot) , will be denoted by $Z^*(G)$. It is easy to see that $Z^*(G)$ is a subgroup of (G, \cdot) .

Corollary 2.4. Suppose $(G, f) = \text{der}_\theta(G, \cdot)$. Then we have

$$\frac{\text{Aut}(G, f)}{Z^*(G)} \hookrightarrow \text{Aut}(G, \cdot).$$

If further, all idempotents of (G, f) are central, then we have

$$\text{Aut}(G, f) \cong C_{\text{Aut}(G, \cdot)}(\theta) \ltimes Z^*(G).$$

Proof. Note that an automorphism $\psi \in \text{Aut}(G, f)$ can be uniquely expressed in the form $\psi = R_a\phi$, so we can define a map $q : \text{Aut}(G, f) \rightarrow \text{Aut}(G, \cdot)$ by $q(R_a\phi) = \phi$. It is easy to check that $(R_a\phi)(R_{a'}\phi') = R_{\phi(a')a}\phi\phi'$. Hence q is a group homomorphism. Clearly, its kernel is $Z^*(G)$, so

$$\frac{\text{Aut}(G, f)}{Z^*(G)} \hookrightarrow \text{Aut}(G, \cdot).$$

Now, let all idempotents of (G, f) be central, so

$$\text{Aut}(G, f) = \{R_a\phi : a \in Z^*(G), \phi \in C_{\text{Aut}(G, \cdot)}(\theta)\}.$$

Define an action of $C_{Aut(G, \cdot)}(\theta)$ on $Z^*(G)$ by the rule $\phi.a = \phi(a)$. One can check that the map

$$\lambda : C_{Aut(G, \cdot)}(\theta) \ltimes Z^*(G) \rightarrow Aut(G, f)$$

with the rule $\lambda(\phi, a) = R_a\phi$ is an isomorphism. This completes the proof.

Corollary 2.5. *Suppose $(G, f) = der_\theta(G, \cdot)$ is a medial polyadic group. Then*

$$Aut(G, f) \cong C_{Aut(G, \cdot)}(\theta) \ltimes Z^*(G).$$

Corollary 2.6. *Let (G, \cdot) be an abelian group and $(G, f) = der^n(G, f)$. Then*

$$Aut(G, f) = Aut(G, \cdot) \ltimes \bar{Z}(G),$$

where $\bar{Z}(G) = \{a \in G : a^{n-1} = 1\}$.

We are ready now, to talk about the structure of automorphisms of a polyadic group in the general form $(G, f) = der_{\theta, b}(G, \cdot)$. Note that θ and b satisfy the conditions of the theorem 1.1.

Theorem 2.7. *Suppose $(G, f) = der_{\theta, b}(G, \cdot)$. Then we have*

$$Aut(G, f) = \{R_a\phi : \phi \in Aut(G, \cdot), f\binom{n}{a} = \phi(b)a, [\theta, \phi] = I_a\},$$

Proof. Let $a \in G$ and $\phi \in Aut(G, \cdot)$ satisfy $f\binom{n}{a} = \phi(b)a$ and $[\theta, \phi] = I_a$. We show that $\psi = R_a\phi$ is an automorphism of (G, f) . For any $x \in G$ and $1 \leq i \leq n-1$, we have

$$\phi\theta^i(x) = a\theta(a) \cdots \theta^{i-1}(a)\theta^i(\phi(x))\theta^{i-1}(a^{-1}) \cdots \theta(a^{-1})a^{-1}.$$

Hence

$$\begin{aligned} R_a\phi(f(x_1^n)) &= R_a\phi(x_1\theta(x_2) \cdots \theta^{n-1}(x_n)b) \\ &= \phi(x_1)\phi(\theta(x_2)) \cdots \phi(\theta^{n-1}(x_n))\phi(b)a \\ &= \phi(x_1)(a\theta(\phi(x_2))a^{-1})(a\theta(a)\theta^2(a)(\phi(x_3))\theta^2(a^{-1})\theta(a^{-1})a^{-1}) \cdots \\ &\quad (a\theta(a) \cdots \theta^{n-2}(a)\theta^{n-1}(\phi(x))\theta^{n-2}(a^{-1}) \cdots \theta(a^{-1})a^{-1})f\binom{n}{a} \\ &= (\phi(x_1)a)(\theta(\phi(x_2))\theta(a)) \cdots (\theta^{n-1}(\phi(x_n))\theta^{n-1}(a))b \\ &= f(R_a\phi(x_1), \dots, R_a\phi(x_n)). \end{aligned}$$

Now, to show that every automorphism of (G, f) has the required form, suppose $(G, g) = der^n(G, \cdot)$. It is clear that

$$T = (\varepsilon, \theta, \theta^2, \dots, \theta^{n-1}, R_{b^{-1}})$$

is an isotopy between (G, f) and (G, g) . So we can do the same argument as in 2.2, to complete the proof.

Lemma 2.8. *Let $(G, f) = \text{der}_{\theta, b}(G, \cdot)$ and suppose $u \in G$ is a central idempotent. Then the map*

$$R_u : \text{der}_{\theta}(G, \cdot) \rightarrow (G, f),$$

is an isomorphism.

Corollary 2.9. *If $(G, f) = \text{der}_{\theta, b}(G, \cdot)$ has a central idempotent, then*

$$\text{Aut}(G, f) \cong \text{Aut}(\text{der}_{\theta}(G, \cdot)).$$

3. HOMOTOPY AND THE STRUCTURE OF HOMOMORPHISMS

Employing the same method as in the section 2, we want to determine the structure a homomorphism between two polyadic groups. We apply the results of this section, to investigate representations of polyadic groups in the section 4.

Lemma 3.1. *Suppose (G, \cdot) and $(H, *)$ are two ordinary groups. Then every homomorphism*

$$\psi : \text{der}^n(G, \cdot) \rightarrow \text{der}^n(H, *)$$

can be uniquely decomposed as $\psi = R_a \phi$ such that

- i- $\phi : (G, \cdot) \rightarrow (H, *)$ is an ordinary homomorphism,
- ii- $a \in C_H(\phi(G))$,
- iii- $a^{n-1} = 1$.

The converse is also true.

Proof. Suppose ψ is given and let $a = \psi(1)$. Then $\psi(1^n) = a$ and so $a^{n-1} = 1$. Now, define $\phi = R_a^{-1}\psi$. We have

$$\begin{aligned} \phi(xy) &= \psi(xy) * a^{-1} \\ &= \psi(x \cdot 1^{n-2} \cdot y) * a^{-1} \\ &= \psi(x) * a^{n-2} * \psi(y) * a^{-1} \\ &= \psi(x) * a^{-1} * \psi(y) * a^{-1} \\ &= \phi(x) * \phi(y). \end{aligned}$$

So, ϕ is an ordinary homomorphism. Further

$$\begin{aligned} \psi(x) &= \psi(1 \cdot x \cdot 1^{n-2}) \\ &= a * \psi(x) * a^{-1}, \end{aligned}$$

so $a * \psi(x) = \psi(x) * a$, which implies that $a \in C_H(\phi(G))$. Conversely, suppose $\psi = R_a \phi$, such that a and ϕ satisfy the above three conditions. We have

$$\begin{aligned} \psi(x_1 x_2 \dots x_n) &= \phi(x_1 x_2 \dots x_n) * a \\ &= \phi(x_1) * \phi(x_2) * \dots * \phi(x_n) * a^n \\ &= \phi(x_1) * a * \phi(x_2) * a * \dots * \phi(x_n) * a \\ &= \psi(x_1) * \psi(x_2) * \dots * \psi(x_n). \end{aligned}$$

This completes the proof.

The following theorem is a generalization of 2.1.

Theorem 3.2. *Suppose (G, \cdot) and $(H, *)$ are two ordinary groups. Then every homotopy*

$$\text{der}^n(G, \cdot) \rightarrow \text{der}^n(H, *)$$

can be decomposed as

$$(L_{a_1} I_{a_1}, L_{a_2} I_{a_1 * a_2}, \dots, L_{a_n} I_{a_1 * \dots * a_n}, R_{a_1 * \dots * a_n}) \circ (R_a, \dots, R_a) \circ (\phi, \dots, \phi),$$

such that

- i- $a_1^n \in H$,
- ii- $\phi : (G, \cdot) \rightarrow (H, *)$ is an ordinary homomorphism,
- iii- $a \in C_H(\phi(G))$,
- iv- $a^{n-1} = 1$.

Proof. Let $T = (\alpha_1, \alpha_2, \dots, \alpha_{n+1})$ be a homotopy from $\text{der}^n(G, \cdot)$ to $\text{der}^n(H, *)$. So, we have

$$\alpha_{n+1}(x_1 x_2 \dots x_n) = \alpha_1(x_1) * \dots * \alpha_n(x_n).$$

Hence, for an arbitrary x and i , we have

$$\begin{aligned} \alpha_{n+1}(x) &= \alpha_{n+1}(1^{i-1} \cdot x \cdot 1^{n-i}) \\ &= \alpha_1(1) * \dots * \alpha_{i-1}(1) * \alpha_i(x) * \alpha_{i+1}(1) * \dots * \alpha_n(1). \end{aligned}$$

Therefore,

$$\alpha_i(x) = (\alpha_1(1) * \dots * \alpha_{i-1}(1))^{-1} \alpha_{n+1}(x) (\alpha_{i+1}(1) * \dots * \alpha_n(1))^{-1}.$$

Now, suppose $\alpha_i(1) = a_i$ and $a_1 * a_2 * \dots * a_n = d$. We have

$$\begin{aligned} \alpha_{n+1}(x_1 x_2 \dots x_n) &= \alpha_1(x_1) * \alpha_2(x_2) * \dots * \alpha_n(x_n) \\ &= \alpha_{n+1}(x_1) * d^{-1} * \alpha_{n+1}(x_2) * d^{-1} * \dots \\ &\quad * \alpha_{n+1}(x_{n-1}) * d^{-1} * \alpha_{n+1}(x_n). \end{aligned}$$

Hence, we have

$$\begin{aligned} \alpha_{n+1}(x_1 x_2 \dots x_n) * d^{-1} &= \alpha_{n+1}(x_1) * d^{-1} * \alpha_{n+1}(x_2) * d^{-1} * \dots \\ &\quad * \alpha_{n+1}(x_{n-1}) * d^{-1} * \alpha_{n+1}(x_n) * d^{-1}. \end{aligned}$$

Therefore, if we let $\psi = R_d^{-1} \alpha_{n+1} = R_a \alpha_{n+1}$, then $\psi : \text{der}^n(G, \cdot) \rightarrow \text{der}^n(H, *)$ is a homomorphism. Now, for any i ,

$$\begin{aligned} \alpha_i(x) &= (\alpha_1(1) * \dots * \alpha_{i-1}(1))^{-1} \alpha_{n+1}(x) (\alpha_{i+1}(1) * \dots * \alpha_n(1))^{-1} \\ &= (a_1 * \dots * a_{i-1})^{-1} * \psi(x) * d * (a_{i+1} * \dots * a_n)^{-1} \\ &= (a_1 * \dots * a_{i-1})^{-1} * \psi(x) * (a_1 * \dots * a_i) \\ &= L_{(a_1 * \dots * a_{i-1})^{-1}} R_{a_1 * \dots * a_i} \psi(x) \\ &= L_{a_i} I_{a_1 * \dots * a_i} \psi(x). \end{aligned}$$

Now, applying lemma 3.1, we obtain the required decomposition for T .

Corollary 3.3. *Suppose $(G, f) = \text{der}_{\theta, b}(G, \cdot)$ and $(H, h) = \text{der}_{\eta, c}(H, *)$ are two n -ary groups. Then every homotopy from (G, f) to (H, h) can be decomposed as the composition of the following homotopies*

$$(\varepsilon, \theta, \theta^2, \dots, \theta^{n-1}, R_b^{-1})$$

$$(\phi, \phi, \dots, \phi)$$

$$(R_a, R_a, \dots, R_a)$$

$$(L_{a_1} I_{a_1}, L_{a_2} I_{a_1 * a_2}, \dots, L_{a_n} I_{a_1 * \dots * a_n}, R_{a_1 * \dots * a_n})$$

$$(\varepsilon, \eta^{-1}, \eta^{-2}, \dots, \eta^{-n-1}, R_c)$$

such that

- i- $a_1^n \in H$,
- ii- $\phi : (G, \cdot) \rightarrow (H, *)$ is an ordinary homomorphism,
- iii- $a \in C_H(\phi(G))$,
- iv- $a^{n-1} = 1$.

Proof. It is clear that $T = (\varepsilon, \theta, \theta^2, \dots, \theta^{n-1}, R_b^{-1})$ is an isotopy from (G, f) to $\text{der}^n(G, \cdot)$ and also $(\varepsilon, \eta^{-1}, \eta^{-2}, \dots, \eta^{-n-1}, R_c)$ is an isotopy from $\text{der}^n(H, *)$ to (H, h) . Now, using 3.3, the result follows.

Corollary 3.4. *Let $\psi : \text{der}_{\theta, b}(G, \cdot) \rightarrow \text{der}_{\eta, c}(H, *)$ be a homomorphism. Then there exists $a \in H$ and an ordinary homomorphism $\phi : (G, \cdot) \rightarrow (H, *)$, such that $\psi = R_a \phi$.*

Proof. The required decomposition for ψ can be obtained by the composition of the first entries of the five homotopies of the previous corollary.

Although there exist some relations between a and ϕ in the previous corollary, we are not able to determine these relations completely. However, in the special case, when $\eta = \varepsilon$ and $c = 1$, we have a sufficient and necessary condition for a and ϕ .

Theorem 3.5. *Let $\psi : \text{der}_{\theta, b}(G, \cdot) \rightarrow \text{der}^n(H, *)$ be a homomorphism. Then there exists $a \in H$ and an ordinary homomorphism $\phi : (G, \cdot) \rightarrow (H, *)$, such that $\psi = R_a \phi$. Further a and ϕ satisfy the following conditions;*

$$a^{n-1} = \phi(b) \quad \text{and} \quad \phi\theta = I_{a^{-1}}\phi.$$

*Conversely, if a and ϕ satisfy the above two conditions, then $\psi = R_a \phi$ is a homomorphism $\text{der}_{\theta, b}(G, \cdot) \rightarrow \text{der}^n(H, *)$.*

Proof. Let $(G, f) = \text{der}_{\theta, b}(G, \cdot)$ and suppose ψ is given. Then by the corollary 3.4, $\psi = R_a \phi$, for some ordinary homomorphism ϕ and some $a \in H$.

We have

$$\begin{aligned}\psi(b) &= \psi(1^{n-1} \cdot b) \\ &= \psi(1)^n,\end{aligned}$$

hence, $a^{n-1} = \phi(b)$. Further,

$$\begin{aligned}\phi(\theta(x))\phi(b) &= \phi(\theta(x)b) \\ &= \psi(\theta(x)b) * a^{-1} \\ &= \psi(f(1, x, 1, \dots, 1)) * a^{-1} \\ &= a * \psi(x) * a^{n-2} * a^{-1} \\ &= a * \phi(x) * a^{n-2}.\end{aligned}$$

So, $\phi\theta = I_{a^{-1}}\phi$. Conversely, suppose $\psi = R_a\phi$, such that a and ϕ satisfy the above mentioned conditions. Now, for all natural number i and all x , we have $\phi(\theta^i(x)) = a^i * \phi(x) * a^{-i}$, so

$$\begin{aligned}\psi(f(x_1^n)) &= \psi(x_1\theta(x_2)\dots\theta^{n-1}(x_n)b) \\ &= \phi(x_1\theta(x_2)\dots\theta^{n-1}(x_n)b) * a \\ &= \phi(x_1) * \phi(\theta(x_2)) * \dots * \phi(\theta^{n-1}(x_n)) * \phi(b) * a \\ &= \phi(x_1) * a * \phi(x_2) * a^{-1} * a^2 * \phi(x_3) * a^{-2} * \dots \\ &\quad * a^{n-1} * \phi(x_n) * a^{-(n-1)} * a^{n-1} * a \\ &= (\phi(x_1) * a) * (\phi(x_2) * a) * \dots * (\phi(x_n) * a) \\ &= \psi(x_1) * \dots * \psi(x_n).\end{aligned}$$

This completes the proof.

There is one more special case which we can determine completely the structure of homomorphisms. In the last theorem of this section, we investigate this special case.

Theorem 3.6. *Let $(G, f) = \text{der}_{\theta, b}(G, \cdot)$ and $(H, h) = \text{der}_{\eta, c}(H, *)$ and suppose further, $(H, *)$ is abelian. Let $\psi : (G, f) \rightarrow (H, h)$ be a homomorphism. Then there exists $a \in H$ and an ordinary homomorphism $\phi : (G, \cdot) \rightarrow (H, *)$, such that $\psi = R_a\phi$, and*

$$h\left(\overset{(n)}{a}\right) = a * \phi(b) \quad \text{and} \quad \phi\theta = \eta\phi.$$

*Conversely, if $a \in H$ and $\phi : (G, \cdot) \rightarrow (H, *)$ is a homomorphism, satisfying the above two conditions, then $\psi = R_a\phi$ is a homomorphism $(G, f) \rightarrow (H, h)$.*

Proof. By the lemma 3.1, and since $(H, *)$ is abelian, every homomorphism $\text{der}^n(G, \cdot) \rightarrow \text{der}^n(H, *)$ can be uniquely decomposed as $R_a\phi$, such that $a \in H$, $a^{n-1} = 1$ and ϕ is an ordinary homomorphism. Hence, by 3.3, every homotopy from (G, f) to (H, h) has the form

$$T = (\varepsilon, \eta^{-1}, \dots, \eta^{-(n-2)}, \varepsilon, R_c) \circ (R_{a_1}, \dots, R_{a_n}, R_{a_1 * \dots * a_n}) \\ \circ (\phi, \dots, \phi) \circ (\varepsilon, \theta, \dots, \theta^{n-1}, R_b^{-1}).$$

Here, $a_1^n \in H$ and ϕ is an ordinary homomorphism. Now, the homomorphisms $\psi : (G, f) \rightarrow (H, h)$ are in one-one correspondence with homotopies which have equal entries. So, we assume that the entries of T are equal. Let $a = a_1$. It is easy to see that the equality of the first and the last entries, implies $a_2 * \dots * a_n * c = \phi(b)$. The equality of the first and the second entries, implies $a_2 = \eta(a)$. Similarly, we have $a_{i+1} = \eta^i(a)$. Now, using $a_2 * \dots * a_n * c = \phi(b)$, we conclude $h\left(\begin{smallmatrix} n \\ a \end{smallmatrix}\right) = a * \phi(b)$. Also, the equality of the first and the second entries, implies $\eta^{-1}\phi\theta = \phi$ and more generally, $\eta^{-i}\phi\theta^i = \phi$. Conversely, if we assume that a and ϕ satisfy the required conditions, we can show that all entries of T are equal, and so $\psi = R_a\phi$ is a homomorphism.

4. REPRESENTATIONS

Suppose (G, f) is an n -ary group. A map $\Lambda : G \rightarrow GL_m(\mathbb{C})$ with the property

$$\Lambda(f(x_1^n)) = \Lambda(x_1)\Lambda(x_2)\dots\Lambda(x_n)$$

is a *representation* of G . The function

$$\chi(x) = \text{Tr } \Lambda(x)$$

is called the corresponding *character* of Λ . The number m is the degree of representation. Note that, Λ is a representation of (G, f) , iff it is an n -ary homomorphism $(G, f) \rightarrow \text{der}^n(GL_m(\mathbb{C}))$.

In [7], representation theory of polyadic groups was studied, but representations dealt with in that paper were considered to have non-empty kernels. The connection of representations of polyadic groups to representations of their retract investigated in that paper. Further, it is shown that, one can reduce the representation theory of polyadic groups to the representation theory of ordinary groups (see [7]). In [12], the second author studied representations of polyadic groups without any assumptions on kernels, i.e. the representations he dealt with in [12], may have empty kernels, as well. It is proved that there is a one-one correspondence between the set of irreducible representations of a polyadic group and the set of irreducible representations of its so called *Post covers*. Using this correspondence, some well-known properties of irreducible characters of finite groups (such as orthogonality of characters and so on), are generalized to finite polyadic groups.

In this section, we can use our knowledge about the structure of homomorphisms of polyadic groups, we just obtained in the section 3, to determine representations of polyadic groups. Applying theorem 3.5, on the case $\Lambda : \text{der}_{\theta,b}(G, \cdot) \rightarrow \text{der}^n(GL_m(\mathbb{C}))$, we obtain the following theorem.

Theorem 4.1. *Let $\Lambda : \text{der}_{\theta,b}(G, \cdot) \rightarrow \text{der}^n(GL_m(\mathbb{C}))$ be a representation. Then we have $\Lambda = R_A \Gamma$, where $A \in GL_m(\mathbb{C})$, $\Gamma : (G, \cdot) \rightarrow GL_m(\mathbb{C})$ is an ordinary representation and we have*

$$A^{n-1} = \Lambda(b) \quad \text{and} \quad \Lambda\theta = I_A^{-1}\Lambda.$$

The converse is also true.

REFERENCES

- [1] A. Borowiec, W. A. Dudek, S. Duplij, *Bi-element representations of ternary groups*, Communications in Algebra **34** (2006), 1651 – 1670.
- [2] W. Dörnte, *Untersuchungen über einen verallgemeinerten Gruppenbegriff*, Math. Z. **29** (1929), 1 – 19.
- [3] W. A. Dudek, *On n -ary group with only one skew element*, Radovi Matematički (Sarajevo), **6** (1990), 171 – 175.
- [4] W. A. Dudek, *Remarks on n -groups*, Demonstratio Math. **13** (1980), 165 – 181.
- [5] W. A. Dudek, K. Glazek, *Around the Hosszú-Gluskin Theorem for n -ary groups*, Discrete Math. **308** (2008), 4861 – 4876.
- [6] W. A. Dudek, J. Michalski, *On a generalization of Hosszú theorem*, Demonstratio Math. **15** (1982), 437 – 441.
- [7] W. Dudek, M. Shahryari, *Representation theory of polyadic groups*, Algebras and Representation Theory, DOI 10.1007/S10468-010-9231-9 (2010).
- [8] K. Glazek, B. Gleichgewicht, *Abelian n -groups*, Colloquia Math. Soc. J. Bolyai 29 Universal Algebra, Esztergom (Hungary) 1977), 321 – 329. (North-Holland, Amsterdam 1982.)
- [9] E. Kasner, *An extension of the group concept*, Bull. Amer. Math. Soc. **10** (1904), 290 – 291.
- [10] A. Marini, V. Shcherbacov, *On autotopies and automorphisms of n -ary linear quasigroups*, Algebra and Discrete Mathematics, **2** (2004), 59 – 83.
- [11] E. L. Post, *Polyadic groups*, Trans. Amer. Math. Soc. **48** (1940), 208 – 350.
- [12] M. Shahryari, *Representation theory of finite polyadic groups*, Submitted.
- [13] V. Shcherbacov, *On structure of finite n -ary medial quasigroups and automorphism groups of these quasigroups*, Quasigroups and Related Systems, **13** (2005), 125 – 156.
- [14] J. D. H. Smith, *An introduction to quasi-groups and their representations*, Chapman & Hall/CRC, Taylor & Francis Group, (2007).
- [15] J. Uşan, *n -groups in the light of the neutral operations*, Monograph, University of Kragujevac, (2005).

DEPARTMENT OF PURE MATHEMATICS, FACULTY OF MATHEMATICAL SCIENCES, UNIVERSITY OF TABRIZ, TABRIZ, IRAN

DEPARTMENT OF PURE MATHEMATICS, FACULTY OF MATHEMATICAL SCIENCES, UNIVERSITY OF TABRIZ, TABRIZ, IRAN

E-mail address: mshahryari@tabrizu.ac.ir